# Motion 12.03: Inner addition/subtraction over intervals

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## 1 Introduction

This paper specifies the operations for inner addition and subtraction over intervals for the forthcoming IEEE interval standard [1].

These operations can be used e. g. for convenient presentation of the solutions of certain interval algebraic equations. For example the inner difference of the intervals $A, B$ is the solution of the equation $B + X = A$ when this solution exists (i. e. when width$(A) \leq$ width$(B)$), or is the solution of the equation $A - X = B$ when the solution exists (i. e. when width$(A) \geq$ width$(B)$). Many applications are related to ranges of monotone functions or the control of accuracy of interval-arithmetic computational results. Some of these applications are briefly outlined in Section 3.

## 2 Inner addition and subtraction over intervals

In real interval-arithmetic inner addition and inner subtraction over two intervals $A = [\underline{a},\ \overline{a}]$, $B = [\underline{b},\ \overline{b}] \in \mathbb{IR}$ are defined as:

$$A +^{-} B = \begin{cases} [\underline{a} + \overline{b},\ \overline{a} + \underline{b}], & \text{if } w(A) \geq w(B), \\ [\overline{a} + \underline{b},\ \underline{a} + \overline{b}], & \text{otherwise.} \end{cases} \tag{1}$$

$$A -^{-} B = \begin{cases} [\underline{a} - \underline{b},\ \overline{a} - \overline{b}], & \text{if } w(A) \geq w(B), \\ [\overline{a} - \overline{b},\ \underline{a} - \underline{b}], & \text{otherwise.} \end{cases} \tag{2}$$

where $w(A) = \overline{a} - \underline{a}$ is the width of $A$.

Remark. Note that (1), (2) are always defined and thus they are "operations" in algebraic sense (and not partial operations).

Inner addition and inner subtraction are related by $A +^- B = A -^- (-B)$, $A -^- B = A +^- (-B)$, where $-B = (-1) * B$.

In accordance to Motion 5 [6] outward digital roundings shall be available for the above operations, that is: $\diamondsuit(A +^- B)$, $\diamondsuit(A -^- B)$.

Remark. The outwardly rounded inner operations $\diamondsuit(A +^- B)$, $\diamondsuit(A -^- B)$ can be defined in the spirit of [43], section 5.6(3), as follows:

`Inner operations --- outward rounding.  There is an operation innerAdditionOut(xx,yy) that returns for any two intervals xx=[l,u] and yy=[l′,u′] the tightest interval containing (the points) l+u′ and u+l′.`

`There is an operation innerSubtractionOut(xx,yy) that returns for any two intervals xx=[l,u] and yy=[l′,u′] the tightest interval containing (the points) l-l′ and u-u′.`

Remark. Exceptional situations, such as $\infty - \infty$ will be treated in accordance with Motion 8 semantics.

Remark. Another type of digital roundings (named inward roundings) will be the subject of a future motion.

# 3 Rationale

In this section we consider two aspects on inner operations: the algebraic one and one for the presentation of functional ranges and computation with such ranges.

The operations for inner addition and subtraction over intervals are mentioned in [43], see p. 37, Section 5.6.(3). Both operations have been used in numerous applications, see e. g. [7]–[40].

## 3.1 Algebraic properties of inner operations

Usually Hukuhara difference [5] is defined in the set of convex bodies $K$ as follows: Given any two sets $A, B \in K$, if there exists a set $X \in K$ satisfying $A = B + X$, then $X = A \ominus B$ is called the *Hukuhara difference* of the sets $A$ and $B$. The Hukuhara difference plays an important role in the theory of convex bodies [45].

The Hukuhara difference can be symbolically expressed as follows:

$$A \ominus B = X \Longleftrightarrow A + X = B. \tag{3}$$

Formula (3) shows that Hukuhara difference is the solution $X$ of $A+X = B$ (whenever existing).

In the special case of one-dimensional intervals $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}] \in \mathbb{IR}$ the Hukuhara difference can be written as:

$$A \ominus B = \begin{cases} [\underline{a} - \underline{b}, \overline{a} - \overline{b}], & \text{if } \ w(A) \geq w(B), \\ \text{not defined}, & \text{otherwise}, \end{cases} \tag{4}$$

wherein $w(A) = \overline{a} - \underline{a}$ is the width of $A$.

Remark. Note that $A \ominus B$ is not an operation in the algebraic sense, but only a partial operation.

The *inner* operations for addition/subtraction of (one-dimensional) intervals from the present motion (1), (2) can be introduced using a similar "algebraic approach". Thus the *inner difference* $A -^{-} B$, $A, B \in I(\mathbb{R})$, is the solution $Z$ of a "linear" equation either of the type $B + Z = A$, or of the type $A - Z = B$ (depending on which one is solvable; one of the equations is always solvable).

For more clarity we shall next denote multiplication by $-1$ as $\neg A = (-1) * A = [-\overline{a}, -\underline{a}]$, resp. subtraction of intervals $A, B$ as $A \neg B = A + (\neg B) = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$.

Remark. Introducing the operation of inner subtraction makes notation $A - B$ vague — is this outer (standard) subtraction or inner? Therefore we shall avoid in the sequel the dubious notation $A - B$ using $A \neg B$ or $A -^{+} B$ for the outer (standard) subtraction and $A -^{-} B$ for the inner one.

Symbolically we have

$$A -^{-} B = X \Longleftrightarrow \begin{cases} B + X = A, & \text{if solution} X \text{exists;} \\ A \neg X = B, & \text{if solution} X \text{exists.} \end{cases} \tag{5}$$

An equivalent way to express the above is:

$$A -^- B = \begin{cases} Y|_{B+Y=A}, & \text{if } \ w(B) \leq w(A); \\ X|_{A \neg X=B}, & \text{if } \ w(A) \leq w(B). \end{cases} \tag{6}$$

Similarly inner addition can be introduced:

$$A +^- B = X \Longleftrightarrow \begin{cases} \neg B + X = A, & \text{if solution} X \text{exists;} \\ \neg A + X = B, & \text{if solution} X \text{exists,} \end{cases} \tag{7}$$

which can be alternatively written as:

$$A +^- B = \begin{cases} Y|_{\neg B+Y=A}, & \text{if } \ w(B) \leq w(A), \\ X|_{\neg A+X=B}, & \text{if } \ w(A) \leq w(B). \end{cases} \tag{8}$$

Note that $w(\neg B) = w(B)$ so that the inequalities $w(B) \leq w(A)$ and $w(\neg B) \leq w(A)$ are equivalent. Inner addition and inner subtraction are related by $A +^- B = A -^- (\neg B)$, $A -^- B = A +^- (\neg B)$ as this easily follows from (6), (8).

**Notation.** For $A, B \in \mathbb{IR}$ denote $A +^+ B = A + B$, $\ A -^+ B = A \neg B = A + (\neg B)$. Then using the binary symbol $\sigma \in \{+, -\}$ we can write $A +^\sigma B$, $A -^\sigma B$.

Remark. In what follows we adopt the end-point notation $A = [a^-,\ a^+]$, $B = [b^-,\ b^+] \in \mathbb{IR}$, which turns out to be very convenient when studying the algebraic properties of inner operations.

Besides the sign functional $\sigma$ we use $\phi : \mathbb{IR} \bigotimes \mathbb{IR} \rightarrow \{+, -\}$, defined as

$$\phi(A, B) = \begin{cases} +, \ \text{if } w(A) \geq w(B); \\ -, \ \text{otherwise}, \end{cases}$$

For $A = [a^-, a^+] \in \mathbb{IR}$ we can define $A +^- B$ using $\phi$ by:

$$A +^- B = [a^{-\gamma} + b^\gamma, a^\gamma + b^{-\gamma}], \ \ \gamma = \phi(A, B), \tag{9}$$

We recall the "join" operation (denoted symbolically "$\vee$") in the special case of real numbers. For $\alpha, \beta \in \mathbb{R}$, the join $\alpha \vee \beta \in \mathbb{R}$ is either the interval $[\alpha, \beta] \in \mathbb{IR}$ or the interval $[\beta, \alpha] \in \mathbb{IR}$ depending on whether $\alpha \leq \beta$ or $\alpha \geq \beta$.

Using join we can write

$$A +^- B = [(a^- + b^+) \vee (a^+ + b^-)].$$

The operation $A -^- B = A +^- (\neg B)$ can be written

$$A -^- B \;=\; [a^{-\gamma} - b^{-\gamma}, a^{\gamma} - b^{\gamma}], \qquad \gamma = \phi(A, B). \tag{10}$$

Using the operation "join", the above can be written

$$A -^- B \;=\; [(a^- - b^-) \vee (a^+ - b^+)].$$

The two operations for addition $+, +^-$ can be considered as one operation in two modes (directions) to be denoted "$+^\theta$", wherein $\theta \in \{+, -\}$, and referred to as "directed addition". For $\theta = +$ the operation $+^\theta$ is the standard (positively directed) addition, "$+$", whereas for $\theta = -$, $+^\theta$ is the nonstandard (negatively directed) addition, "$+^-$". The directed addition $+^\theta$ can be expressed:

$$A +^\theta B = [(a^- + b^{-\theta}) \vee (a^+ + b^\theta)].$$

**Rules for algebraic transformations.** Below we present some properties of directed/inner addition.

Commutativity of inner addition. For $A, B \in \mathbb{IR}$ we have $A +^- B = B +^- A$.

*Conditional associativity of directed addition.* Directed addition is conditionally associative in the sense that for each triple $A, B, C \in I(\mathbb{R})$ and each pair $\theta_1, \theta_2 \in \{+, -\}$, there exist another pair $\theta_3, \theta_4 \in \{+, -\}$, such that

$$(A +^{\theta_1} B) +^{\theta_2} C = A +^{\theta_3} (B +^{\theta_4} C).$$

Moreover, $\theta_3, \theta_4$ are simple functions of the widths of the intervals and can be easily computed.

$X = [0, 0] = 0$ is the unique neutral element with respect to inner addition $+^-$, that is for every $A \in \mathbb{IR}$

$$A = X +^- A = A +^- X \Longleftrightarrow X = [0, 0].$$

Note that $A +^- (\neg A) = 0$, thus every element $A \in \mathbb{IR}$ has unique inverse w. r. t. "$+^-$", and this is the element $\neg A = [-a^+, -a^-]$.

Outer addition is commutative and associative but has no inverse, whereas inner addition is commutative, not associative and has inverse. Considered together, as one "directed" operation in two different modes, we can say that this directed operation is conditionally acossiative. So both modes complement each other.

For $p \in \mathbb{R}$ define $\sigma(p) = \{+, \text{ if } p \geq 0;\ -, \text{ if } p < 0\}$.

Quasidistributive law: For $A \in \mathbb{IR}, p, q \in \mathbb{R}$ and $*$ multiplication by scalars

$$(p + q) * A = p * A +^{\sigma(p)\sigma(q)} q * A, \tag{11}$$

Remark. Note the equality relation in (11); recall that the corresponding law formulated in classic (outer) operations only gives inclusion: $(p+q)*C \subseteq p*C + q*C$, whereas now we have $(p + q) * C = p * C +^{\sigma(pq)} q * C$.

## 3.2 Inner operations and monotone functions

Let $X \in \mathbb{IR}$ and $f, g$ be two continuous functions defined on $x \in X$.

For the functional ranges $f(X) = \{f(x) \mid x \in X\},\ g(X) = \{g(x) \mid x \in X\}$ we have $(f + g)(X) \subseteq f(X) + g(X)$.

Moreover, we have $(f + g)(X) = f(X) + g(X)$, if $f, g$ equally monotone and this is true for arbitrary equally monotone functions $f, g$.

This observation can be used to define the operation addition of two intervals $A, B \in \mathbb{IR}$ as follows:

Definition. Given two intervals $A, B \in \mathbb{IR}$ take any two equally monotone functions $f, g$ defined on $X$, s. t. $f(X) = A,\ g(X) = B$. We then define the sum of the intervals $A, B \in \mathbb{IR}$ as $A + B = (f + g)(X)$.

The above definition is correct, since $(f + g)(X)$ depends only on the choice of $A, B$.

Example 1. If $X = [0, 1],\ A = [a^-, a^+],\ B = [b^-, b^+]$, we can choose the functions $f, g$ to be the monotone increasing (isotone) linear functions defined on $[0, 1]$ by:

$$f^+(\xi) = (1-\xi)a^- + \xi a^+, \quad g^+(\xi) = (1-\xi)b^- + \xi b^+,$$

or the monotone decreasing (antitone) linear functions defined on $[0,1]$ as:

$$f^-(\xi) = (1-\xi)a^+ + \xi a^-, \quad g^-(\xi) = (1-\xi)b^+ + \xi b^-.$$

Remark. In practice, for smooth functions $f, g$ and for small interval argument $X$ in "half" of the situations $f, g$ are *equally* monotone functions and we have $A + B = (f+g)(X)$ (showing the usefulness of the operation interval addition). Now the question is what can be done in the other "half" of the cases when $f, g$ are two *differently* monotone functions. In this situation denoting $f(X) = A,\ g(X) = B$, we only have $(f+g)(X) \subseteq f(X) + g(X) = A+B$, but the inclusion can be "rough". Typical example is $f(x) = x,\ g(x) = -x$, then we have $0 = \{x + (-x) \mid x \in X\} \subseteq X + (-X) = X - X$, with $\omega(X - X) = 2\omega(X)$, showing how rough an inclusion can be.

Following arguments similar to the ones for the above definition of addition $A + B = (f+g)(X)$ we can proceed as follows.

Definition. Given $A, B \in \mathbb{IR}$, take any two differently monotone functions $f, g$ s. t. $f(X) = A,\ g(X) = B$ and $f + g$ is monotone. Define "inner addition" by means of $A +^- B = (f+g)(X)$.

Obviously, $(f+g)(X)$ depends only on the choice of $A, B$ so the above definition is correct.

The above can be summarized as follows.

Denote $CM(T)$ the set of continuous monotone functions on $T \in \mathbb{IR}$.
For $f \in CM(T)$ denote $\tau_f = \tau(f;T) \in \{+,-\}$, where

$$\tau(f;T) = \begin{cases} +, & \text{if } f \text{ is isotone in } T; \\ -, & \text{if} f \text{ is antitone in } T. \end{cases}$$

For $f, g \in CM(T)$, the equality $\tau_f = \tau_g$ means, that both $f, g$ are isotone or both are antitone in $T$; $\tau_f = -\tau_g$ means that one function is isotone and the other is antitone.

Proposition 1. ([33], [34]) Let $f, g \in CM(T)$. Then for every $X \subseteq T$ we have:

i) $f + g \in CM(T)$ implies $(f+g)(X) = f(X) +^{\tau_f \tau_g} g(X)$,

ii) $f - g \in CM(T)$ implies $(f-g)(X) = f(X) -^{-\tau_f \tau_g} g(X)$.

The application of the above proposition can be illustrated by means of the following examples [8].

Example 2. Consider the problem of finding exact interval expressions for Taylor series of elementary functions, such as $\exp, \ln, \cos, \sin$, whenever X belongs to some interval within certain domain. For $X \geq 0$ we have, using familiar interval addition: $\exp(X) = 1 + X/1! + X^2/2! + X^3/3! + X^4/4! + ....$

However, this expression gives overestimation for other values of $X$. In such cases inner operations can be helpful. Applying the monotonicity Proposition 1 for ranges of $X$ such that $-1 \leq X < 0$, we obtain:

$$\exp(X) = 1 + X/1! +^{-} X^2/2! + X^3/3! +^{-} X^4/4! + ... \ , -1 \leq X < 0.$$

Example 3. Similarly, using familiar interval addition/subtraction, we have $\ln(1 + X) = X - X^2/2 + X^3/3 - X^4/4 + ...$, $0 \leq X \leq 1$.

However, for $-1 < X \leq 0$ the above formula is not exact. Based on Proposition 1, using inner subtraction we obtain:

$$\ln(1 + X) = X -^{-} X^2/2 + X^3/3 -^{-} X^4/4 + ..., \ -1 < X \leq 0.$$

The order of the execution of operations in the above examples is from left to right. It is assumed that the ranges $X^n$ are exact in the specified ranges for $X$.

A computer algebra system having additional information for the domains of the interval arguments can perform automatically the resulting expressions.

When dealing with complicated expressions, the process of finding narrow/exact interval bounds can be done automatically by means of Proposition 1. The automatization process has been nicely described and used in [9] (there inner operations for multiplication/division are used as well). The process is based on the automatic check of the monotonicity of the (sub)expressions involved, starting from the most inner subexpressions, similarly to the process of automatic differentiation.

Other applications known to us (listed incompletely). Baker Kearfott has a Fortran implementation of inner interval addition/subtraction [4]. A. Neumeier uses inner interval operations for efficient constraint propagation in solving global optimization problems, in COCONUT and in GloptLab, see formulae (25), (26) and Proposition 14.2 of [41], see also [43], section 5.6(3). R. Alt, J.-L. Lamotte and V. Kreinovich make use of inner interval arithmetic in [2]. L. Stefanini uses inner operation in the context of fuzzy set theory [46]. V. Nesterov makes use of inner operations in [39], [40].

Remark. Many applications make use of variants of inner operations, e. g. one variant based on Hukuhara difference (that is, partial operation), such as [43], section 5.6(3), and another one leading to nonstandard (Kaucher) intervals, such as formulae (25), (26) of [41].

Remark. A correspondence between inner operations and the exist/forall modes in modal interval arithmetic has been studied, e. g. in [3], [35].

# 4 Mid-rad presentation

The mid-rad presentation of intervals adds important further insight to the application of inner interval-arithmetic operations.

Denoting the midpoint and the radius of $A = [a^-, a^+] \in \mathbb{IR}$ resp. by $a'$ and $a''$, we have

$$a' = (a^- + a^+)/2, \quad a'' = (a^+ - a^-)/2.$$

The form $A = (a'; a'')$ is called mid-rad presentation. Conversely we have

$$a^- = a' - a'', \quad a^+ = a' + a''.$$

For (auter) addition and subtraction we have

$$\begin{aligned} A + B &= (a' + b'; a'' + b''), \\ A \neg B &= (a' - b'; a'' + b''). \end{aligned}$$

For inner addition and subtraction we have

$$\begin{aligned} A +^- B &= (a' + b'; |a'' - b''|), \\ A -^- B &= (a' - b'; |a'' - b''|). \end{aligned}$$

Remark. The above formulae illuminate the interval operations when radii are small, which corresponds to the “approximate number” aspect of intervals. Outer and inner operations have same midpoints, which means that the main value is same; however, inner operations have smaller radii, that is error bounds. Thus mid-rad presentations help us in understanding the idea of "interval arithmetic operations in two modes". Considered as approximate numbers, inner and outer operations produce same main values but with different error bounds. In particular inner operations can be used to treat the so-called "dependency problem".

Using multiplication by $-1$ outer/inner addition is representable by means of outer/inner subtraction, that is $A +^{\theta} B = A -^{\theta} (\neg B)$, $\theta = \pm$.

Multiplication by $-1$ in mid-rad presentation is

$\neg A = (-1) * A = (-1) * (a'; a'') = (a'; |a''|),\ A \in \mathbb{IR}$.

More generally, we have for $\gamma \in \mathbb{R}$

$$\gamma * A = \gamma * (a'; a'') = (\gamma a'; |\gamma| a''). \tag{12}$$

Recall that multiplication by scalars in end-point presentation has the form:

$$\gamma * A = \begin{cases} [\gamma a^-, \gamma a^+], & \text{if } \gamma \geq 0, \\ [\gamma a^+, \gamma a^-], & \text{if } \gamma < 0. \end{cases} \tag{13}$$

Remark. Note that the components midpoint/radius in (12) are "separated", which is not the case of multiplication by scalars in end-point presentation (13). Thus mid-rad presentation allows to reduce certain classes of linear interval problems to linear numerical problems [38] .

# 5 Relation to Kaucher/modal arithmetic

To illuminate the relation between inner interval operations and the operations in Kaucher/modal arithmetic we shall consiser an analogy from real arithmetic [47].

As we know the introduction of negative numbers is a rather new event in the long history of mathematics. As G. Birkhoff notes in [48]: We should not forget that zero and negative numbers were among the last to be accepted. The primary use of negative numbers (and zero) is to make the equation $A + X = B$ always solvable, i. e. to make the additive monoid $(\mathbb{R}^+, +)$ of nonnegative numbers a group. The isomorphic extension (embedding) of a commutative monoid into a group is now a common mathematical tool [54]. However, before that mathematicians like Diophantus from Alexandria also used negative numbers in a primitive way, similar to the way now this is done in interval analysis.

To give a brief idea of inner interval operations it is instructive to compare the algebraic properties of standard intervals to those of nonnegative numbers. Recall the useful operation "$\ominus$" in the additive semigroup $(\mathbb{R}^+, +)$ of nonnegative numbers: $A \ominus B$ is the solution $X$ of $A + X = B$, if $X$ exists, and is the solution $X$ of $B + X = A$, if $X$ exists. Of course $A \ominus B = |A - B|$, but we do not have negative numbers $-B$ in $(\mathbb{R}^+, +)$, and hence we cannot write $A \ominus B = A + (-B)$.

The inner operations for standard intervals can be defined similarly.

According to formula (7) the inner difference $A -^- B$ is the solution $X$ of B + X = A, if such a solution $X$ exists, and is the solution $X$ of $A + (-1) * X = B$ provided $X$ exists, if both solutions exist, they coincide.

A commutative semigroup can be embedded in a group if and only if it is cancellative [55]. A commutative monoid $(M, +)$ is cancellative and thus embeddable in a group. We recall that a commutative monoid (M,+) is cancellative if for all $a, b, c \in M$, $a + b = a + c$ always implies $b = c$.

The algebraic structure $(\mathbb{IR}, +)$ of (standard) intervals with addition is an commutative monoid, and thus embeddable in a group; this has been noticed already by M. Warmus [53] and T. Sunaga [52], [50]. The new elements involved are the so-called improper intervals which together with the standard ones constitute the group of generalized (modal/Kaucher) intervals.

The basic operations (addition and multiplication) and relation inclusion have to be isomorphically extended for the group elements, preserving the important property of inclusion isotonicity. An isomorphic extension of multiplication (addition is trivial) from the semigroup $\mathbb{IR}$ to the group of modal intervals based on set-theoretic arguments (i.e. preserving inclusion isotonicity) has been correctly done by H.-J. Ortolf [51] and E. Kaucher [49].

More about the relation between inner interval operations and modal/Kaucher intervals can be found in [35].

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