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Algebraic Operations on the Space of Hausdorff Continuous Interval Functions

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We show that the operations addition and multiplication on the set $C(\Omega)$ of all real continuous functions on $\Omega \subseteq \mathbb{R}^n$ can be extended to the set $\mathbb{H}(\Omega)$ of all Hausdorff continuous interval functions on Ω in such a way that the algebraic structure of $C(\Omega)$ is preserved, namely, $\mathbb{H}(\Omega)$ is a commutative ring with identity. The operations on $\mathbb{H}(\Omega)$ are defined in three different but equivalent ways. This allows us to look at these operations from different points of view as well as to show that they are naturally associated with the Hausdorff continuous functions.

1. Introduction

The set $\mathbb{H}(\Omega)$ of all Hausdorff continuous interval functions appears naturally in the context of Hausdorff approximations of real functions. The concept of Hausdorff continuity of interval functions generalizes the concept of continuity of real function in such a way that many essential properties of the usual continuous real functions are preserved. Not least this is due to the fact that the Hausdorff continuous functions assume real (point) values on a dense subset of the domain and are completely determined by these values. It is well-known that the set $C(\Omega)$ of all continuous real functions defined on a subset Ω of $\mathbb{R}^n$ is a commutative ring with respect to the point-wise defined addition and multiplication of functions. Hence the natural question: is it possible to extend the algebraic operations on $C(\Omega)$ to the set $\mathbb{H}(\Omega)$ of all Hausdorff continuous functions defined on Ω in a way which preserves the algebraic structure, that is, the set of $\mathbb{H}(\Omega)$ is a commutative ring with respect to the extended operations? We show in this paper that the answer to this question is positive. Furthermore, we give three different but equivalent ways of defining algebraic operations on $\mathbb{H}(\Omega)$ with the required properties. Namely, (i) through the point-wise interval operations, (ii) by using an extension property of the Hausdorff continuous functions, (iii) through the order convergence structure on $\mathbb{H}(\Omega)$.

2. General Setting

The real line is denoted by $\mathbb{R}$ and the set of all finite real intervals by $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\}$. Given an interval $a = [\underline{a}, \overline{a}] = \{x : \underline{a} \leq x \leq \overline{a}\} \in \mathbb{IR}$, $w(a) = \overline{a} - \underline{a}$ is the width of a, while $|a| = \max\{|\underline{a}|, |\overline{a}|\}$ is the modulus of a. An interval a is called proper interval, if $w(a) > 0$ and point interval, if $w(a) = 0$. Identifying $a \in \mathbb{R}$ with the point interval $[a, a] \in \mathbb{IR}$, we consider $\mathbb{R}$ as a subset of $\mathbb{IR}$. We denote by $\mathbb{A}(\Omega)$ the set of all locally bounded interval-valued functions defined on an arbitrary set $\Omega \subseteq \mathbb{R}^n$. The set $\mathbb{A}(\Omega)$ contains the set $\mathcal{A}(\Omega)$ of all locally bounded real functions defined on Ω. Recall that a real function or an interval-valued function f defined on Ω is called locally bounded if for every $x \in \Omega$ there exist $\delta > 0$ and $M \in \mathbb{R}$ such that $|f(y)| < M$, $y \in B_\delta(x)$, where $B_\delta(x) = \{y \in \Omega : \|x - y\| < \delta\}$ denotes the open δ-neighborhood of x in Ω.

Let D be a dense subset of Ω. The mappings

$$I(D, \Omega, \cdot),\ S(D, \Omega, \cdot) : \mathbb{A}(D) \longrightarrow \mathcal{A}(\Omega)$$

defined for $f \in \mathbb{A}(D)$ and $x \in \Omega$ by

$$I(D, \Omega, f)(x) = \sup_{\delta > 0} \inf\{f(y) : y \in B_\delta(x) \cap D\},$$
$$S(D, \Omega, f)(x) = \inf_{\delta > 0} \sup\{f(y) : y \in B_\delta(x) \cap D\},$$

are called lower and upper Baire operators, respectively. The mapping

$$F : \mathbb{A}(D) \longrightarrow \mathbb{A}(\Omega),$$

called *graph completion operator*, is defined by

$$F(D, \Omega, f)(x) = [I(D, \Omega, f)(x), S(D, \Omega, f)(x)], \quad x \in \Omega,\ f \in \mathbb{A}(D).$$

In the case when $D = \Omega$ the sets D and Ω will be omitted, thus we write $I(f) = I(\Omega, \Omega, f)$, $S(f) = S(\Omega, \Omega, f)$, $F(f) = F(\Omega, \Omega, f)$.

Definition 1. A function $f \in \mathbb{A}(\Omega)$ is S-continuous, if $F(f) = f$.

Definition 2. A function $f \in \mathbb{A}(\Omega)$ is Hausdorff continuous (H-continuous), if $g \in \mathbb{A}(\Omega)$ with $g(x) \subseteq f(x)$, $x \in \Omega$, implies $F(g)(x) = f(x)$, $x \in \Omega$.

Theorem 1 ([1, 7]). For every $f \in \mathbb{A}(\Omega)$ the functions $F(I(S(f)))$ and $F(S(I(f)))$ are H-continuous.

The following theorem states an essential property of the continuous functions which is preserved by the H-continuity, [1].

Theorem 2. *Let $f, g \in \mathbb{H}(\Omega)$. If there exists a dense subset D of Ω such that $f(x) = g(x)$, $x \in D$, then $f(x) = g(x)$, $x \in \Omega$.*

H-continuous functions are also similar to the usual continuous real functions in that they assume point values everywhere on Ω except for a set of first Baire category. More precisely, it is shown in [1] that for every $f \in \mathbb{H}(\Omega)$ the set $W_f = \{x \in \Omega : w(f(x)) > 0\}$ is of first Baire category and f is continuous on $\Omega \setminus W_f$. Since a finite or countable union of sets of first Baire category is also a set of first Baire category we have:

Theorem 3. *Let the set Ω be open and let $\mathcal{F}$ be a finite or countable set of H-continuous functions. Then the set*

$$D_{\mathcal{F}} = \{x \in \Omega : w(f(x)) = 0,\ f \in \mathcal{F}\} = \Omega \setminus \bigcup_{f \in \mathcal{F}} W_f \tag{1}$$

is dense in Ω and all functions $f \in \mathcal{F}$ are continuous on $D_{\mathcal{F}}$.

The graph completion operator is inclusion isotone: i) w. r. t. the functional argument, that is, if $f, g \in \mathbb{A}(D)$, where D is dense in Ω, then

$$f(x) \subseteq g(x),\ x \in D \implies F(D, \Omega, f)(x) \subseteq F(D, \Omega, g)(x), \quad x \in \Omega,$$

and, ii) w. r. t. the set D in the sense that if D_1 and D_2 are dense subsets of Ω and $f \in \mathbb{A}(D_1 \cup D_2)$, then

$$D_1 \subseteq D_2 \implies F(D_1, \Omega, f)(x) \subseteq F(D_2, \Omega, f)(x), \quad x \in \Omega. \tag{2}$$

In particular, (2) implies that for any dense subset D of Ω and $f \in \mathbb{A}(\Omega)$ we have

$$F(D, \Omega, f)(x) \subseteq F(f)(x), \quad x \in \Omega. \tag{3}$$

Let $f \in \mathbb{A}(\Omega)$. For every $x \in \Omega$ the value of f is an interval $[\underline{f}(x), \overline{f}(x)] \in \mathbb{IR}$. Hence, f can be written in the form $f = [\underline{f}, \overline{f}]$ where $\underline{f}, \overline{f} \in \mathcal{A}(\Omega)$ and $\underline{f}(x) \leq \overline{f}(x),\ x \in \Omega$. The lower and upper Baire operators as well as the graph completion operator of an interval-valued function f can be represented in terms of $\underline{f}$ and $\overline{f}$, namely, for every dense subset D of Ω: $I(D, \Omega, f) = I(D, \Omega, \underline{f})$, $S(D, \Omega, f) = S(D, \Omega, \overline{f})$, $F(D, \Omega, f) = [I(D, \Omega, \underline{f}),\ S(D, \Omega, \overline{f})]$.

3. Interval Operations and Operations on $\mathbb{H}(\Omega)$

The familiar operations of addition and multiplication on the set of real intervals are defined for $[\underline{a}, \overline{a}], [\underline{b}, \overline{b}] \in \mathbb{IR}$ as follows [2]:

$$[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = \{a + b : a \in [\underline{a}, \overline{a}], b \in [\underline{b}, \overline{b}]\} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}],$$

$$[\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}] = \{ab : a \in [\underline{a}, \overline{a}], b \in [\underline{b}, \overline{b}]\} = [\min\{\underline{ab}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{ab}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}].$$

The operations for interval functions are defined point-wise in the usual way:

$$(f + g)(x) = f(x) + g(x), \qquad (f \times g)(x) = f(x) \times g(x). \tag{4}$$

Example 1. Consider the functions $f, g \in \mathbb{H}(\mathbb{R})$ given by

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ [0,1], & \text{if } x = 0 \\ 1, & \text{if } x > 0, \end{cases} \qquad g(x) = \begin{cases} 0, & \text{if } x < 0 \\ [-1,0], & \text{if } x = 0 \\ -1, & \text{if } x > 0. \end{cases}$$

Using (4) we have

$$(f+g)(x) = \begin{cases} 0, & \text{if } x < 0 \\ [-1,1], & \text{if } x = 0 \\ 0, & \text{if } x > 0. \end{cases}$$

Example 1 shows that the point-wise sum of H-continuous functions is not necessarily an H-continuous function. Hence the significance of the following two theorems.

Theorem 4 ([2]). *If the interval functions $f, g \in \mathbb{A}(\Omega)$ are S-continuous, then the functions $f+g$ and $f \times g$ are also S-continuous.*

Theorem 5. *Let $f, g \in \mathbb{H}(\Omega)$.*

(a) There exists a unique function $p \in \mathbb{H}(\Omega)$ satisfying the inclusion $p(x) \subseteq (f+g)(x)$, $x \in \Omega$.

(b) There exists a unique function $q \in \mathbb{H}(\Omega)$ satisfying the inclusion $q(x) \subseteq (f \times g)(x)$, $x \in \Omega$.

Proof. We will prove only (a) because (b) can be proved in a similar way. The existence of the function p follows from Theorem 1. Indeed, both functions $F(I(S(f+g)))$ and $F(S(I(f+g)))$ satisfy the required inclusion. To prove the uniqueness let us assume that $p_1, p_2 \in \mathbb{H}(\Omega)$ both satisfy the inclusion in (a). Consider the set $D_{fg} = \{x \in \Omega : w(f(x)) = w(g(x)) = 0\}$. Obviously $w((f+g)(x)) = 0$ for $x \in D_{fg}$. Therefore, due to the assumed inclusions we have $p_1(x) = p_2(x) = f(x) + g(x)$, $x \in D_{fg}$. According to Theorem 3 the set D_{fg} is dense in Ω. Using that the functions p_1 and p_2 are H-continuous it follows from Theorem 2 that $p_1 = p_2$. □

Now we define the operations $\oplus$ and $\otimes$ as follows.

Definition 3. Let $f, g \in \mathbb{H})(\Omega)$. Then $f \oplus g$ is the unique Hausdorff continuous function satisfying the inclusion $(f \oplus g)(x) \subseteq (f+g)(x)$, $x \in \Omega$; $f \otimes g$ is the unique Hausdorff continuous function satisfying the inclusion $(f \otimes g)(x) \subseteq (f \times g)(x)$, $x \in \Omega$.

The existence of both $f \oplus g$ and $f \otimes g$ is guaranteed by Theorem 5.

It is important to note that the values of $f \oplus g$ and $f \otimes g$ at the points where both operands assume interval values can not be determined point-wise, i.e., from the values of f and g at these points. This is illustrated by the following example.

Example 2. Consider the functions $f, g \in \mathbb{H}(\mathbb{R})$ given by

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0 \\ [-1,1], & \text{if } x = 0, \end{cases} \qquad g(x) = \begin{cases} \cos(1/x), & \text{if } x \neq 0 \\ [-1,1], & \text{if } x = 0. \end{cases}$$

We have

$$(f \oplus g)(x) = \begin{cases} \sqrt{2}\cos(1/x + \pi/4), & \text{if } x \neq 0, \\ [-\sqrt{2}, \sqrt{2}], & \text{if } x = 0. \end{cases}$$

Clearly $(f \oplus g)(0)$ can not be obtained just from the values $f(0)$ and $g(0)$.

According to Theorem 1, for any $f, g \in \mathbb{H}(\Omega)$ the functions $F(S(I(f+g)))$ and $F(I(S(f+g)))$ are Hausdorff continuous. Moreover, these functions satisfy the inclusions

$$F(S(I(f+g)))(x) \subseteq (f+g)(x), \quad F(I(S(f+g))(x) \subseteq (f+g)(x), \quad x \in \Omega.$$

Therefore they both coincide with $f \oplus g$. Hence we have

$$f \oplus g = F(I(S(f+g))) = F(S(I(f+g)).$$

In a similar way we obtain

$$f \otimes g = F(I(S(f \times g))) = F(S(I(f \times g))).$$

One can immediately see from the above representations that if f and g are usual continuous real functions we have $f \oplus g = f + g$ and $f \otimes g = f \times g$. Hence $\oplus$ and $\otimes$ extend the operations of addition and multiplication on $C(\Omega)$. A further motivation for considering these operations is in the fact that the algebraic structure of $C(\Omega)$ is preserved as stated in the next theorem.

Theorem 6. *The set $\mathbb{H}(\Omega)$ is a commutative ring with identity with respect to the operations $\oplus$ and $\otimes$.*

Proof. The commutative laws for both $\oplus$ and $\otimes$ follow immediately from Definition 3. It is also obvious that the additive identity is the constant zero function while the multiplicative identity is the constant function equal to 1. We will show now the existence of the additive inverse. Let $f = [\underline{f}, \overline{f}] \in \mathbb{H}(\Omega)$. Consider the function $g \in \mathbb{H}(\Omega)$ given by $g(x) = [-\overline{f}(x), -\underline{f}(x)]$, $x \in \Omega$. Clearly we have

$$0 \in f(x) + g(x), \quad x \in \Omega.$$

Then, according to Definition 3, $f \oplus g$ is the constant zero function.

The proof of the associative and distributive laws is an easy application of the techniques derived in the next section and will be proved there. $\square$

4. Extension Property and an Alternative Definition of the Operations on $\mathbb{H}(\Omega)$

Let D be a dense subset of Ω. Extending a function f defined on D to Ω while preserving its properties (e.g. linearity, continuity) is an important issue in functional analysis. Recall that if f is continuous on D it does not necessarily have a continuous extension on Ω. Hence the significance of the next theorem which was proved in [3].

Theorem 7. *Let $\varphi \in \mathbb{H}(D)$, where D is a dense subset of Ω. Then there exists a unique $f \in \mathbb{H}(\Omega)$ such that $f(x) = \varphi(x)$, $x \in D$. Namely, $f = F(D, \Omega, \varphi)$.*

For every two functions $f, g \in \mathbb{H}(\Omega)$ denote

$$D_{fg} = \{x \in \Omega : w(f(x)) = w(g(x)) = 0\}.$$

As was shown already the point-wise sum and product of H-continuous functions is not always H-continuous. However, the restrictions of f, g, $f + g$ and $f \times g$ on the set D_{fg} are all real continuous functions, see Theorem 3. As usual, these restrictions are denoted by $f|_{D_{fg}}$, $(f+g)|_{D_{fg}}$, etc. Using that the set D_{fg} is dense in Ω, the following definition of the operations $\oplus$ and $\otimes$ is suggested.

Definition 4. Let $f, g \in \mathbb{H}(\Omega)$. Then

(a) $f \oplus g$ is the unique H-continuous extension of $(f + g)|_{D_{fg}}$ on Ω given by Theorem 7, that is, $f \oplus g = F(D_{fg}, \Omega, f + g)$.

(b) $f \otimes g$ is the unique H-continuous extension of $(f \times g)|_{D_{fg}}$ on Ω given by Theorem 7, that is, $f \otimes g = F(D_{fg}, \Omega, f \times g)$.

In order to justify the use of the notations $\oplus$ and $\otimes$ let us immediately prove that Definitions 3 and 4 are equivalent. Indeed, using property (3) and the fact that $f + g$ is S-continuous, see Theorem 4, we have

$$F(D_{fg}, \Omega, f + g)(x) \subseteq F(f + g)(x) = (f + g)(x), \quad x \in \Omega.$$

Hence $F(D_{fg}, \Omega, f + g)$ is the unique Hausdorff continuous function satisfying the inclusion required in Definition 3, which implies that $F(D_{fg}, \Omega, f+g)$ is the sum of f and g according to Definition 3. In a similar way one can show that $F(D_{fg}, \Omega, f \times g)$ is the product of f and g according to Definition 3. Therefore Definitions 3 and 4 are equivalent.

Definition 4 is particularly useful for evaluating arithmetical expressions involving more than two operands since one can evaluate the expression on a set where all operands assume point values and then extend the resulting function to Ω. We will explain the procedure in detail. Let $E(+, \times, z_1, z_2, ..., z_k)$ be

an expression involving the operations $+$ and $\times$ and k operands. Let $\mathcal{F} = \{f_1, f_2, ..., f_k\} \subset \mathbb{H}(\Omega)$. Then the functions

$$E(+, \times, f_1, f_2, ..., f_k) \quad \text{and} \quad E(\oplus, \otimes, f_1, f_2, ..., f_k) \tag{5}$$

are both well defined, the first one being S-continuous, the second one being H-continuous. According to Theorem 3 the set $D_{\mathcal{F}}$ given by (1) is dense in Ω. Then a simple connection between the functions (5) is given in the next theorem, which is proved easily using the definition of the operations and the extension property.

Theorem 8. *For any set of functions $\mathcal{F} = \{f_1, f_2, ..., f_k\} \subset \mathbb{H}(\Omega)$ and arithmetical expression $E(+, \times, z_1, z_2, ..., z_k)$ we have*

$$F(D_{\mathcal{F}}, \Omega, E(+, \times, f_1, f_2, ..., f_k)) = E(\oplus, \otimes, f_1, f_2, ..., f_k)$$

As an application of Theorem 8 we show that the associative and distributive laws of a ring hold true on $\mathbb{H}(\Omega)$ with respect to the operations $\oplus$ and $\otimes$. This completes the proof of Theorem 6.

Proof of the associative and distributive laws on $\mathbb{H}(\Omega)$. Let $f, g, h \in \mathbb{H}(\Omega)$ and let $D_{\mathcal{F}}$ be the dense subset of Ω given by (1) with $\mathcal{F} = \{f, g, h\}$. Since the values of f, g and h on the set $D_{\mathcal{F}}$ are all real numbers (point intervals), the functions f, g and h satisfy on $D_{\mathcal{F}}$ the associative and distributive laws with respect to the operations $+$ and $\times$. Then using Theorem 8 we have

$$(f \oplus g) \oplus h = F(D_{\mathcal{F}}, \Omega, (f + g) + h) = F(D_{\mathcal{F}}, \Omega, f + (g + h)) = f \oplus (g \oplus h),$$
$$(f \otimes g) \otimes h = F(D_{\mathcal{F}}, \Omega, (f \times g) \times h) = F(D_{\mathcal{F}}, \Omega, f \times (g \times h)) = f \otimes (g \otimes h),$$
$$\begin{aligned}(f(x) \oplus g(x)) \otimes h(x) &= F(D_{\mathcal{F}}, \Omega, (f + g) \times h) \\ &= F(D_{\mathcal{F}}, \Omega, f \times h + g \times h) = f \otimes h \oplus g \otimes h,\end{aligned}$$

which shows that both associative laws as well as the distributive law hold true. □

5. Definition of the Operations on $\mathbb{H}(\Omega)$ through the Order Convergence Structure

Partial order can be defined for intervals in different ways. Here we consider the partial order on $\mathbb{IR}$ given by

$$[\underline{a}, \overline{a}] \leq [\underline{b}, \overline{b}] \iff \underline{a} \leq \underline{b},\ \overline{a} \leq \overline{b}\ . \tag{6}$$

The partial order on $\mathbb{H}(\Omega)$ which is induced by (6) in a point-wise way, that is, for $f, g \in \mathbb{H}(\Omega)$

$$f \leq g \iff f(x) \leq g(x), \quad x \in \Omega, \tag{7}$$

is naturally associated with $\mathbb{H}(\Omega)$. For example, it was shown in [1] that the set $\mathbb{H}(\Omega)$ is Dedekind order complete with respect to this order.

The order convergence of sequences on a poset is defined through the partial order.

Definition 5. Let P be a poset with a partial order $\leq$. A sequence $(f_n)_{n\in\mathbb{N}}$ on P is said to order converge to $f \in P$ if there exist on P an increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ and a decreasing sequence $(\beta_n)_{n\in\mathbb{N}}$ such that $\alpha_n \leq \xi_n \leq \beta_n$, $n \in \mathbb{N}$, and $f = \sup_{n\in\mathbb{N}} \alpha_n = \inf_{n\in\mathbb{N}} \beta_n$.

It is well-known that in general the order convergence on a poset is not topological in the sense that there is no topology with class of convergent sequences exactly equal to the class of the order convergent sequences. In particular this is the case of $\mathbb{H}(\Omega)$ with respect to the partial order (7). However, the order convergence induces on $\mathbb{H}(\Omega)$ the structure of the so-called FS sequential convergence space. See [6] for the definition and further details on FS sequential convergence spaces and convergence (filter) spaces.

The concept of Cauchy sequence in general can not be defined within the realm of sequential convergence only but rather using the stronger concept of a convergence space [6]. It is shown in [4] that the order convergence structure on $C(\Omega)$ is induced by a convergence space and we have the following characterization of the Cauchy sequences. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence on $C(\Omega)$. Then

$$(f_n)_{n\in\mathbb{N}} \text{ is Cauchy } \iff \begin{cases} \text{There exists a decreasing sequence} \\ (\beta_n)_{n\in\mathbb{N}} \text{ on } C(\Omega) \text{ such that } \inf \beta_n = 0 \\ \text{and } f_m - f_k \leq \beta_n,\ m,k \geq n,\ n \in \mathbb{N}. \end{cases} \tag{8}$$

It is also shown in [4] that the convergence space completion of $C(\Omega)$ is $\mathbb{H}(\Omega)$. More precisely, we have the following theorem.

Theorem 9. *(i) For every Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ on $C(\Omega)$ there exists $f \in \mathbb{H}(\Omega)$ such that $f_n \to f$.*

(ii) For every $f \in \mathbb{H}(\Omega)$ there exists a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ on $C(\Omega)$ such that $f_n \to f$. Moreover, the sequence $(f_n)_{n\in\mathbb{N}}$ can be selected to be either increasing or decreasing.

Definition 6. Let $f, g \in \mathbb{H}(\Omega)$ and let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be the Cauchy sequences on $C(\Omega)$ existing in terms of Theorem 9, that is, we have $f_n \to f$, $g_n \to g$. Then $f \oplus g$ is the order limit of $(f_n + g_n)_{n\in\mathbb{N}}$ and $f \otimes g$ is the order limit of $(f_n \times g_n)_{n\in\mathbb{N}}$.

Let us first note that the order limits stated in Definition 6 do exist. Indeed, since $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are Cauchy sequences on $C(\Omega)$ one can see from (8) that their sum $(f_n + g_n)_{n\in\mathbb{N}}$ and their product $(f_n \times g_n)_{n\in\mathbb{N}}$ are Cauchy sequences. Hence according to Theorem 9 (i) they both order converge.

To establish the consistency of the Definition 6 we need to show that $f \oplus g$ and $f \otimes g$ do not depend on the particular choice of the sequences. Let $(f_n^{(1)})_{n\in\mathbb{N}}$,

$(f_n^{(2)})_{n\in\mathbb{N}}$, $(g_n^{(1)})_{n\in\mathbb{N}}$, $(g_n^{(2)})_{n\in\mathbb{N}}$ be Cauchy sequences on $C(\Omega)$ such that $f_n^{(i)} \to f$, $g_n^{(i)} \to g$, $i = 1, 2$. We shall show that the sequences $(f_n^{(1)} + g_n^{(1)})_{n\in\mathbb{N}}$ and $(f_n^{(2)} + g_n^{(2)})_{n\in\mathbb{N}}$ converge to the same limit, resp. the sequences $(f_n^{(1)} \times g_n^{(1)})_{n\in\mathbb{N}}$ and $(f_n^{(2)} \times g_n^{(2)})_{n\in\mathbb{N}}$ converge to the same limit. Denote by $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ the trivial mixtures of the sequences $(f_n^{(1)})_{n\in\mathbb{N}}$, $(f_n^{(2)})_{n\in\mathbb{N}}$ and, respectively, $(g_n^{(1)})_{n\in\mathbb{N}}$, $(g_n^{(2)})_{n\in\mathbb{N}}$, that is, $f_{2n-1} = f_n^{(1)}$, $f_{2n} = f_n^{(2)}$, $g_{2n-1} = g_n^{(1)}$, $g_{2n} = g_n^{(2)}$. In an FS sequential convergence space the trivial mixture of sequences converging to the same limit also converges to that limit [6]. Hence we have $f_n \to f$, $g_n \to g$. It is easy to see that $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are Cauchy sequences. Hence $(f_n + g_n)_{n\in\mathbb{N}}$ is also Cauchy sequence, which implies that $(f_n + g_n)_{n\in\mathbb{N}}$ converges on $\mathbb{H}(\Omega)$, see Theorem 9 (i). Therefore, $(f_n^{(1)} + g_n^{(1)})_{n\in\mathbb{N}}$ and $(f_n^{(2)} + g_n^{(2)})_{n\in\mathbb{N}}$, being subsequences of the order convergent sequence $(f_n + g_n)_{n\in\mathbb{N}}$ order converge to the same limit. In a similar way we prove that $(f_n^{(1)} \times g_n^{(1)})_{n\in\mathbb{N}}$ and $(f_n^{(2)} \times g_n^{(2)})_{n\in\mathbb{N}}$ converge to the same limit.

Theorem 10. *Definition 3 and Definition 6 are equivalent.*

Proof. Let $f, g \in \mathbb{H}(\Omega)$ and let h_1 be their sum according to Definition 3 while h_2 is their sum according to Definition 6. We shall show that $h_1 = h_2$. It follows from Theorem 9(ii) that we can select increasing sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ on $C(\Omega)$ such that $f_n \to f$, $g_n \to g$ or equivalently, $f = \sup\limits_{n\in\mathbb{N}} f_n$, $g = \sup\limits_{n\in\mathbb{N}} g_n$. Then, according to Definition 6, h_2 is the order limit of the increasing sequence $(f_n + g_n)_{n\in\mathbb{N}}$, that is, $h_2 = \sup\limits_{n\in\mathbb{N}}(f_n + g_n)$. On the other hand,

$$f_n(x) + g_n(x) \leq \underline{f}(x) + \underline{g}(x) \leq h_1(x), \quad x \in \Omega.$$

Therefore, $h_2 \leq h_1$. In a similar way by using decreasing sequences we prove that $h_2 \geq h_1$. Hence $h_1 = h_2$.

The proof of the equivalence of definitions of multiplication is done using a similar approach but is technically more complicated and will be omitted. □

6. Conclusion

H-continuous functions are special interval-valued functions which are commonly used in practice, e. g. histograms are such functions. However, in order to avoid arithmetic operations with intervals, such functions are traditionally treated by means of real-valued functions, such as, e. g. semi-continuous functions. A difficulty in using H-continuous functions is that, if we add two H-continuous functions that have interval values at same argument using pointwise interval arithmetic, then we may obtain as a result an interval function which is not H-continuous. In this work we define addition and multiplication so that the set of H-continuous functions is closed under these operations. We

show that the arithmetic operations addition and multiplication on the set of all Hausdorff continuous interval functions preserve the algebraic structure of the set of real continuous functions, which is a commutative ring with identity. Three different but equivalent definitions are presented, which allow to look at these operations from different points of view as well as to show that they are naturally associated with the Hausdorff continuous functions. These results make H-continuous functions an attractive tool in real analysis and provide a bridge between real and interval analysis.

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