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Towards an Axiomatization of Interval Arithmetic

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In this paper intervals are viewed as approximate real numbers. A revised formula for interval multiplication of generalized intervals is given. This formula will be useful for further axiomatization of interval arithmetic and relevant implementations within computer algebra systems. Relations between multiplication of numbers and multiplication of errors are discussed.

1 Introduction

The concept of *approximate number* or (real one-dimensional) *interval* is a key concept in numerical analysis and mathematical modeling. By an approximate number one usually understands a real number plus/minus an (upper bound for the) *error*. Approximate numbers can be conveniently presented in mid-point/radius form, writing $A = (a'; a'')$, wherein a' is the mid-point and $a'' \geq 0$ is the radius. In this presentation we have e. g. for addition of two intervals $A + B = (a'; a'') + (b'; b'') = (a' + b'; a'' + b'')$. Point intervals are of the form $(a'; 0)$, centred ones (errors) are of the form $(0; a'')$. Using addition in mid-point/radius form we have the simple presentation $A = (a'; a'') = (a'; 0) + (0; a'')$. This presentation corresponds to the symbolic form $A = a' \pm a''$ used in engineering sciences. Approximate numbers generalize the concept of number. Hence, it is natural to extend all familiar arithmetic operations and relations for numbers in the set of approximate numbers. This is one of the subjects of interval analysis and, more specifically, of interval arithmetic. A natural setting is generalized interval arithmetic [1]–[4], where the restriction $a'' \geq 0$ is dropped. Indeed, nonnegative errors do not form a group with respect to addition, they form a monoid. However, the monoid of errors is commutative and obeys the cancellation law. So it embeds nicely in a group, in the way we embed nonnegative numbers. If we want to compute with errors, it is most natural to work in a group where every equation $a + x = b$ possesses a solution. Now, when the monoid of errors is embeded in a group, new elements appear — "negative" errors. The negative errors plus the familiar nonnegative ones form the set of *generalized* errors. We have to extend the arithmetic operations and relation in the set of generalized errors. In this way new algebraic systems appear and their properties have to be studied.

The system of (one-dimensional) generalized intervals with addition and multiplication has been investigated quite fully in the works of E. Kaucher [1]. However, the system of generalized interval vectors with addition and multiplication by scalars — so-called quasivector space — has been only recently examined [3], [4]. And, there is still not enough knowledge about the interaction between the above mentioned systems. Undoubtedly such studies will be of benefit for the development of CAS's and their application to problems involving approximate, stochastic, fuzzy etc. numbers or (certain particular cases of convex) sets. The challenge here is to make CAS able to handle conditional formulae, such as the formulae for multiplication of intervals and to possibly simplify these formulae as much as possible.

2 Operations over errors and approximate numbers

Operations over errors. Let us consider the operation multiplication of errors (centred intervals) in the generalized case when errors may have negative values. Let us denote this operation by "$^\times$". Isotonicity with respect to inclusion is a basic property of interval operations that is crucial for applications, so we should require "$^\times$" to be inclusion isotone. In the simple case of centred intervals this means that we should have $(0; a) \subseteq (0; b) \Longrightarrow (0; c)^\times(0; a) \subseteq (0; c)^\times(0; b)$ for any error $(0; c)$, or, equivalently $a \leq b \Longrightarrow c^\times a \leq c^\times b$ for all real a, b, c. This property should hold for generalized errors, that is for the case when the errors a and b take possibly negative values as well (as we know such a property does not hold for the familiar multiplication). As found by Kaucher, the definition of multipliplication so that inclusion isotonicity holds, is:

$$a^\times b = \begin{cases} ab, & \text{if } a \geq 0,\ b \geq 0, \\ -ab, & \text{if } a \leq 0,\ b < 0, \\ 0, & \text{if } a > 0,\ b < 0 \text{ or } a < 0,\ b > 0. \end{cases} \tag{1}$$

Some examples: $(-2)^\times(-3) = -6$, $2^\times(-3) = 0$. This multiplication has some nice properties. In addition to being inclusion isotone, we have e. g. $-(a^\times b) = (-a)^\times(-b)$. The operation "$^\times$" is not distributive over addition, so $(\mathbb{R}, +, ^\times)$ is not a ring, but if we define a multiplication by $a \cdot b = \mathrm{sign}(a)\mathrm{sign}(b)(|a|^\times|b|)$, then the system $(\mathbb{R}, +, \cdot)$ becomes a ring.

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Consider now the situation with multiplication of errors by scalars. Denote this operation by "$*$", so that $\gamma * (0; a)$ means the result of multiplying the error $(0; a)$ by the scalar γ. In the proper case (in the monoid of nonnegative errors) we have: $\gamma * (0; a) = (0; |\gamma| a)$, showing that an error does not change its sign when multiplied by a negative number. Thus errors are multiplied by scalars according to the rule: $\gamma * a = |\gamma| a$. Naturally this operation is inclusion isotone, that is $a \le b \Longrightarrow \gamma * a \le \gamma * b$, as $a \le b \Longrightarrow |\gamma| a \le |\gamma| b$. Same formula $\gamma * a = |\gamma| a$ is used to extend the definition of "$*$" over the set $\mathbb{R}^n$ of generalized (positive or negative) n-tuples of errors. For example: $-2 * (1, 2, 1) = (2, 4, 2)$; $-1 * (-1, 2, -2) = (-1, 2, -2)$. We see that multiplication by -1 (negation) coincides with identity. We have $a + (-1) * a = a + a = 2 * a \ne 0$.To change the signs of the vector's components one should use the opposite operator (and not negation), i. e. $\mathrm{opp}(-1, 2, -2) = (1, -2, 2)$.

Now we may ask if $(\mathbb{R}^n, +, \mathbb{R}, *)$ is a vector space. The answer is no (as the second distributive law does not hold) — it is a centred quasivector space [4]. However, if we take a multiplication by scalar "$\cdot$" defined by $\gamma \cdot a = \{|\gamma| a,\ \text{if}\ \gamma \ge 0; |\gamma| \mathrm{opp}(a),\ \text{if}\ \gamma < 0\}$, then $(\mathbb{R}^n, +, \mathbb{R}, \cdot)$ is a vector space. Thus, if we suitably re-define the operation multiplication by scalars, then the space becomes linear! Note that quasilinear spaces appear in a similar context in the study of stochastic numbers, convex bodies, and more specifically, zonotopes [5].

Operations over approximate numbers. Once we have studied the properties of errors we can turn to the study of the properties of approximate numbers. It is easy to check that n-tuples of approximate numbers form a quasivector space w. r. t. multiplication by scalars. More precisely, if we denote by Q^n the set of all n-tuples of approximate numbers (n-dimensional interval vectors in the language of interval arithmetic), then $(Q^n, +, \mathbb{R}, *)$ satisfies all axioms of a vector space but the second distributive law. Instead the quasidistributive law $(\alpha + \beta) * c = \alpha * c + \beta * c$, $\alpha \ge 0$, $\beta \ge 0$ holds true. It has been recently proved, that every quasivector space is a direct product of a vector space and a centred quasivector space [4]. This result throws new light on the problem showing that the mid-point/radius presentation is a natural form for the presentation of approximate numbers. Within this presentation the mid-points constitute a vector space and the radii (errors) a centred quasivector space.

As a simple example, consider a linear system with exact (real) coefficients in the matrix and interval right-hand side, and look for the familiar algebraic solution. Symbolically we write the system in the form $A * x = b$, and look for an interval vector x satisfying the system. Equivalently in mid-point/radius form we write $A * (x'; x'') = (b'; b'')$. Due to $A * (x'; x'') = A * ((x'; 0) + (0; x'')) = A * (x'; 0) + A * (0; x'')$, this reduces to the systems $A * (x'; 0) = (b'; 0)$ and $A * (0; x'') = (0; b'')$, briefly $A * x' = b'$ and $A * x'' = b''$. Using that the space of mid-points is linear, the first system becomes the familiar linear system $Ax' = b'$. Using that the space of errors is a centred quasivector space, we obtain $|A| x'' = b''$; here the "quasivector" multiplication by scalar has been translated into the familiar multiplication by scalar so the problem is formulated in well-known terms. We believe that this process can be made automatic within a CAS.

The formula for multiplication of generalized (one-dimensional) approximate numbers in mid-point/radius form involves several condition cases. Below we give a revised formula (cf. formula A.3 in [2]) :

$$a \times b = \begin{cases} (a'b' + \sigma(a')\sigma(b')a''b''; |a'|b'' + |b'|a''), & \kappa(a) \le 1,\ \kappa(b) \le 1; \\ (b' + \sigma(b')\sigma(a'')b'') * (a'; a''), & \kappa(a) > 1 \ge \kappa(b) \text{ or } \kappa(a) \ge \kappa(b) > 1, a''b'' \ge 0; \\ (a' + \sigma(a')\sigma(b'')a'') * (b'; b''), & \kappa(b) > 1 \ge \kappa(a) \text{ or } \kappa(b) \ge \kappa(a) > 1, a''b'' \ge 0; \\ 0, & \kappa(a) \ge 1,\ \kappa(b) \ge 1, a''b'' \le 0; \end{cases} \tag{2}$$

wherein σ is the sign functional and κ is the relative error in $a = (a'; a'')$ defined by: $\kappa(a) = |a''|/|a'|,\quad a' \ne 0$. In formula (2) the intermediate two cases are written as multiplication by scalars; in fact these two cases can be viewed as a single case by a, b interchanging places.

3 Conclusion

The study of the algebraic properties of approximate numbers is important for the development of specific software tools as part of computer algebra systems. In particular, computer algebra systems should be able do deal conveniently with approximate numbers using mid-point/radius presentation. It is important to have suitable formulae for multiplication of errors and approximate numbers like (1), (2) and to be able to translate automatically symbolic expressions using such multiplications into symbolic expressions using familiar multiplications for numbers.

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