On the presentation of ranges of monotone functions using interval arithmetic

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Abstract The paper is devoted to the presentation of ranges of monotone functions of one variable by means of extended interval arithmetic structures. The concept of directed interval is introduced which is an extension of the concept of normal interval and a corresponding "directed interval arithmetic" is briefly considered. A closed relation between the directed interval arithmetic and the interval arithmetic using an extended set of operations over normal intervals is demonstrated. Some applications of the directed interval arithmetic to computing (directed) ranges of monotone functions are considered.

1 Introduction

In a previous paper we define the concept of directed range of a monotone and continuous function and derive formulae for the computation of the directed range of a sum, difference, product and quotient of two monotone functions in terms of directed ranges of the operands (see [15], Prop. 9.). For this purpose the generalized interval arithmetic introduced in [17] has been used. However, it is pointed out in [15] that the same goal can be achieved when using an equivalent generalization based on the concept of directed interval. Here we briefly introduce a relevant arithmetic for such directed intervals and demonstrate its potential use for the presentation of directed ranges of monotone functions. The concept of directed interval seems to be useful for a better comprehension and easy interpretation of certain theoretical results; however it can be also easily implemented into corresponding software modules computing ranges of functions (see e.g. [2] for similar modules).

A *directed range* of a monotone and continuous function a over its interval domain $T = [t_1, t_2]$ is a couple consisting of the range $a(T) = \{a(t)|t \in T\}$ of a (which is a normal interval) and a binary variable containing additional information for the type of monotonicity of a. The type of monotonicity of a determines the direction into which the range $a(T)$ is traced when the argument t of a varies in its interval domain T. Indeed, if a is isotone (nondecreasing) in T then the interval $a(T)$ is traced from left to right whenever t traces T from left to right; alternatively $a(T)$ is traced from right to left if a is antitone (nonincreasing) in T; we thus speak of a plus-type or minus-type range. A directed range can be represented either in the form of a directed interval $[A; \pm] = [a^-, a^+; \pm]$ with $A = [a^-, a^+] \in I(R)$, or in the form of an ordered couple $[a_1, a_2] \in R^2$ of real numbers called generalized intervals [6]–[9], [15], [17]. In the latter case the binary information regarding

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the direction of the interval can be encoded by the order of the endpoints: increasing order means plus type; decreasing order means minus type. Denoting the directed range of a monotone function a over T by $a[T]$ and the type of monotonicity of a on T by $\tau(a;T)$ we can symbolically express $a[T]$ either as the directed interval $a[T] = [a(T); \tau(a;T)]$ or as the generalized interval $a[T] = [a(t_1), a(t_2)]$. If a is isotone on T then the directed range $a[T]$ corresponds to the directed interval of plus type $a[T] = [a(T); +]$ or to a proper interval $[a(t_1), a(t_2)]$ from $I(R)$; if a is antitone on T then $a[T]$ corresponds to a directed interval of minus type $a[T] = [a(T); -]$ or to an improper (irregular) interval (if not degenerated) $a[T] = [a(t_1), a(t_2)]$ with $a(t_1) \geq a(t_2)$.

The interval arithmetic based on generalized intervals is well developed; here we briefly consider an arithmetic for directed intervals. We consider only the abstract case of real endpoints. The practical situation involving machine (floating-point) endpoints and relevant directed roundings requires considerations of inclusion relations and corresponding computational rules; this situation will be considered in a forthcoming paper.

In the next section we briefly introduce the necessary prerequisit. The interval arithmetic structure $\mathcal{M} = (I(R), +, \times, +^-, \times^-)$ based on the set of two familiar arithmetic operations $+, \times$ and two nonstandard operations $+^-, \times^-$ over the set of normal intervals $I(R)$ [10]–[15] is presented using the "plus-minus" techniques for notation of the interval end-points [11], [5], [15]. In section 3 we introduce the interval arithmetic for directed intervals. In section 4 we consider the application of directed interval arithmetic for the presentation of ranges of monotone functions of one variable.

2 Presentation of ranges using normal intervals

Throughout the paper we denote by Λ the set consisting of the symbols "plus" and "minus", $\Lambda = \{+, -\}$. These symbols may have various meanings according to the particular situation: they may refer to the type of the endpoint (left or right), to the type of an interval operation (standard or nonstandard), to the type of a directed interval (plus or minus type) etc.

A normal (proper) interval $[a, b], a \leq b$, is a compact set on the real line R defined by $[a, b] = \{x \mid a \leq x \leq b\}$. The set $\{[a, b] \mid a, b \in R, a \leq b\}$ of all intervals is denoted by $I(R)$. The left end-point of $A \in I(R)$ is denoted by a^- or A^-, and the right end-point by a^+ or A^+, so that $A = [a^-, a^+] = [A^-, A^+]$. Thus a^s (or A^s), with $s \in \Lambda = \{+, -\}$, denotes the left or the right end-point of $A \in I(R)$ depending on the value of s. We define the product st for $s, t \in \Lambda$ by setting $++ = -- = +$, $+- = -+ = -$, so that $a^{++} = a^{--} = a^+$ etc.

Denote the set of intervals containing zero by $Z = \{A \in I(R) \mid 0 \in A\} = \{A \mid a^- \leq 0 \text{ and } a^+ \geq 0\}$; the elements of Z will be called Z-intervals. The set of intervals which do not contain zero is $I(R) \setminus Z = \{A \in I(R) \mid 0 \notin A\}$; such intervals are called zero-free intervals. Define a sign functional $\sigma : I(R) \setminus Z \to \Lambda$, by means of $\sigma(A) = \{+, \text{ if } a^- > 0;\ -, \text{ if } a^+ < 0\}$.

The interval arithmetic $\mathcal{S} = (I(R), +, \times, /, \subseteq)$ [1], [16], [18]–[21] consists of the set

$I(R)$ together with a relation for inclusion $\subseteq$ and the basic operations addition $+ : I(R) \bigotimes I(R) \to I(R)$, multiplication $\times : I(R) \bigotimes I(R) \to I(R)$ and inversion (reciprocal value) $/ : I(R) \setminus Z \to I(R)$, defined by

$$
\begin{aligned}
A \subseteq B &\iff (b^- \le a^-) \text{ and } (a^+ \le b^+), \quad \text{for } A, B \in I(R)\ , && (1)\\
A + B &= [a^- + b^-, a^+ + b^+], \quad \text{for } A, B \in I(R)\ , && (2)\\
A \times B &= \begin{cases} [a^{-\sigma(B)} b^{-\sigma(A)}, a^{\sigma(B)} b^{\sigma(A)}], & \text{for } A, B \in I(R) \setminus Z\ , \\ [a^{\delta} b^{-\delta}, a^{\delta} b^{\delta}], \ \delta = \sigma(A), & \text{for } A \in I(R) \setminus Z,\ B \in Z\ , \\ [a^{-\delta} b^{\delta}, a^{\delta} b^{\delta}], \ \delta = \sigma(B), & \text{for } A \in Z,\ B \in I(R) \setminus Z, \end{cases} && (3)\\
A \times B &= [\min\{a^- b^+, a^+ b^-\}\ ,\ \max\{a^- b^-, a^+ b^+\}], \ \text{ for } A, B \in Z\ , && (4)\\
1\ /\ B &= [1/b^+, 1/b^-],\ B \in I(R) \setminus Z. && (5)
\end{aligned}
$$

In the special case when A is a degenerate interval of the form $A = [a, a] = a$, we have $A \times B = a \times B = [ab^{-\sigma(a)}, ab^{\sigma(a)}] = \{[ab^-, ab^+],\ \text{if } a \ge 0;\ [ab^+, ab^-],\ \text{if } a < 0\}$. For $a = -1$ we have $(-1) \times B = -B = -[b^-, b^+] = [-b^+, -b^-]$. The operations subtraction $A - B$ and division A/B are defined in $\mathcal{S}$ as composite operations by

$$
\begin{aligned}
A - B &= A + (-1) \times B = A + (-B) = [a^- - b^+, a^+ - b^-],\ \text{for } A, B \in I(R), && (6)\\
A/B &= A \times (1/B) = \begin{cases} [a^{-\sigma(B)}/b^{\sigma(A)}, a^{\sigma(B)}/b^{-\sigma(A)}], & \text{for } A, B \in I(R) \setminus Z, \\ [a^{-\delta}/b^{-\delta}, a^{\delta}/b^{-\delta}], \ \delta = \sigma(B), & \text{for } A \in Z, B \in I(R) \setminus Z\ . \end{cases} && (7)
\end{aligned}
$$

The operation inversion $1/B$ in $\mathcal{S}$ can not be composed just by means of the operations $+$ and $\times$ and therefore has to be assumed as one of the basic operations in $\mathcal{S}$. The operations $+, -, \times, /$ in $\mathcal{S}$ defined by (2)–(4), (6)–(7) satisfy the relations: $A * B = \{a * b \mid a \in A, b \in B\}$, $* \in \{+, -, \times, /\}$, which provide a basis for important applications.

From algebraic and practical point of view the structure $\mathcal{S}$ is incomplete. In order to obtain a complete structure we introduce two additional operations $+^-, \times^-$ which turn $\mathcal{S}$ into a powerful interval-arithmetic structure $(I(R), +, +^-, \times, \times^-, \subseteq)$. The additional (nonstandard) interval arithmetic operations $+^-$, $\times^-$ in $I(R)$ (cf. [10]–[15]) are defined by

$$
\begin{aligned}
A +^- B &= [a^{-\gamma} + b^{\gamma}, a^{\gamma} + b^{-\gamma}], \quad \text{for } A, B \in I(R), && (8)\\
A \times^- B &= \begin{cases} [a^{\sigma(B)\varepsilon} b^{-\sigma(A)\varepsilon}, a^{-\sigma(B)\varepsilon} b^{\sigma(A)\varepsilon}], & \text{for } A, B \in I(R) \setminus Z, \\ [a^{-\delta} b^{-\delta}, a^{-\delta} b^{\delta}], \ \delta = \sigma(A), & \text{for } A \in I(R) \setminus Z,\ B \in Z, \\ [a^{-\delta} b^{-\delta}, a^{\delta} b^{-\delta}], \ \delta = \sigma(B), & \text{for } A \in Z,\ B \in I(R) \setminus Z, \\ [\ \max\ \{a^- b^+, a^+ b^-\},\ \min\ \{a^- b^-, a^+ b^+\}], & \text{for } A, B \in Z, \end{cases} && (9)
\end{aligned}
$$

wherein the sign variables $\gamma, \varepsilon \in \Lambda$ are chosen in such a way that the intervals involved in the right-hand sides are elements of $I(R)$, that is $a^{-\gamma} + b^{\gamma} \le a^{\gamma} + b^{-\gamma}$, $a^{\sigma(B)\varepsilon} b^{-\sigma(A)\varepsilon} \le a^{-\sigma(B)\varepsilon} b^{\sigma(A)\varepsilon}$. From these inequalities we can explicitly express γ, ε as follows. Define

$$
\begin{aligned}
\omega(A) &= a^+ - a^-, \quad \text{for } A \in I(R),\\
\chi(A) &= a^{-\sigma(A)}/a^{\sigma(A)} = \{a^-/a^+ \ \text{ if } \sigma(A) = +;\ a^+/a^- \ \text{ if } \sigma(A) = -\},\ \text{for } A \in I(R) \setminus Z,
\end{aligned}
$$

and the sign operators $\phi : I(R) \bigotimes I(R) \to \Lambda$ and $\psi : (I(R) \setminus Z) \bigotimes (I(R) \setminus Z) \to \Lambda$ by

$$\begin{aligned}
\phi(A,B) &= \mathrm{sign}(\omega(A) - \omega(B)) = \{+, \ \text{if } \omega(A) \geq \omega(B); \quad -, \ \text{otherwise}\},\\
\psi(A,B) &= \mathrm{sign}(\chi(A) - \chi(B)) = \{+, \ \text{if } \chi(A) \geq \chi(B); \quad -, \ \text{otherwise}\}.
\end{aligned}$$

Using that for $A, B \in I(R) \setminus Z$ the inequalities $\chi(A) \geq \chi(B)$ and $a^{\sigma(B)} b^{-\sigma(A)} \leq a^{-\sigma(B)} b^{\sigma(A)}$ are equivalent we see that γ, ε in (8), (9) can be defined as $\gamma = \phi(A,B), \ \ \varepsilon = \psi(A,B)$.

The elements $-A = [-a^+, -a^-]$ and $1/A = [1/a^+, 1/a^-]$ are inverse with respect to the operations $+^-$ and $\times^-$, that is $A +^- (-A) = 0, \ \ A \times^- (1/A) = 1$. The following composite operations can be defined:

$$\begin{aligned}
A -^- B &= A +^- (-B) = [a^{-\gamma} - b^{-\gamma}, a^{\gamma} - b^{\gamma}], \ \text{ for } A, B \in I(R),\\
A/^- B &= A \times^- (1/B) = \begin{cases} [a^{\sigma(B)\varepsilon}/b^{\sigma(A)\varepsilon}, a^{-\sigma(B)\varepsilon}/b^{-\sigma(A)\varepsilon}], & \text{for } A, B \in I(R) \setminus Z,\\ [a^{-\delta}/b^{\delta}, a^{\delta}/b^{\delta}], \ \ \delta = \sigma(B), & \text{for } A \in Z, \ B \in I(R) \setminus Z. \end{cases}
\end{aligned}$$

where $\gamma = \phi(A,B), \ \ \varepsilon = \psi(A,B)$. We denote the system $(I(R), +, +^-, \times, \times^-, \subseteq)$ by $\mathcal{M}$. The algebraic properties of $\mathcal{M}$ are well studied (see [3]–[5], [10]–[15]); they incorporate and extend the properties of $\mathcal{S}$. The meaning of the nonstandard operations becomes transparent when considering the arithmetic operations for directed intervals and when applying them to computation of directed ranges. We end this section by recalling the presentation of ranges of monotone functions using the interval arithmetic $\mathcal{M}$.

Denote by $CM(T)$ the set of all continuous and monotone functions on $T \in I(R)$. For a function $f \in CM(T)$ denote

$$\tau(f;T) = \begin{cases} +, & \text{if } f \text{ is isotone in } T;\\ -, & \text{if } f \text{ is antitone in } T. \end{cases}$$

Then for $f, g \in CM(T)$, the relation $\tau(f;T) = \tau(g;T)$ means that both functions are isotone or both are antitone in T; $\tau(f;T) = -\tau(g;T)$ means that one of the functions is isotone and the other is antitone. The following proposition holds true [14].

Proposition 1. For $f, g \in CM(T)$ and $X \subseteq T$:

$$\begin{aligned}
f + g \in CM(T) \Longrightarrow (f+g)(X) &= \begin{cases} f(X) + g(X), & \text{if } \tau(f;T) = \tau(g;T),\\ f(X) +^- g(X), & \text{if } \tau(f;T) = -\tau(g;T); \end{cases}\\
f - g \in CM(T) \Longrightarrow (f-g)(X) &= \begin{cases} f(X) -^- g(X), & \text{if } \tau(f;T) = \tau(g;T),\\ f(X) - g(X), & \text{if } \tau(f;T) = -\tau(g;T); \end{cases}
\end{aligned}$$

In addition to the above assumptions, if f, g do not change sign in T, then

$$\begin{aligned}
fg \in CM(T) \Longrightarrow (fg)(X) &= \begin{cases} f(X) \times g(X), & \text{if } \tau(|f|;T) = \tau(|g|;T),\\ f(X) \times^- g(X), & \text{if } \tau(|f|;T) = -\tau(|g|;T); \end{cases}\\
\left.\begin{array}{l} f/g \in CM(T),\\ g(x) \neq 0, x \in T \end{array}\right\} \Longrightarrow (f/g)(X) &= \begin{cases} f(X)/^- g(X), & \text{if } \tau(|f|;T) = \tau(|g|;T),\\ f(X)/g(X), & \text{if } \tau(|f|;T) = -\tau(|g|;T). \end{cases}
\end{aligned}$$

Example 1. Denote $\exp(-X) = \{\exp(-x) | x \in X\}$, $\mathrm{arctg}(X) = \{\mathrm{arctg} x | x \in X\}$. Using Proposition 1 we obtain for the range of $h(x) = \exp(-x) + \mathrm{arctg}(x)$ the simple expression

$$h(X) = \exp(-X) +^- \mathrm{arctg}(X)$$

for any $X \in I(R), 0 \notin X$; obviously standard interval arithmetic is unable to provide an exact interval arithmetic expression for $h(X)$ using the ranges of $\exp(-x)$ and $\mathrm{arctg}(x)$.

3 Directed interval arithmetic

An ordered triple of the form $\mathbf{A} = [a^-, a^+; \alpha]$, where a^-, a^+ are reals such that $a^- \leq a^+$, and $\alpha \in \Lambda$, will be further referred as a *directed interval*. We shall also present $\mathbf{A}$ as an ordered couple of the form $\mathbf{A} = [A; \alpha]$ with $A \in I(R), \alpha \in \Lambda$. The sign variable α in $\mathbf{A} = [a^-, a^+; \alpha]$ is called *type* or *direction* of the directed interval $\mathbf{A}$, and is denoted by $\tau(\mathbf{A})$; according to the value of $\alpha = \tau(\mathbf{A})$, a directed interval $\mathbf{A} = [a^-, a^+; \alpha]$ can be of plus or of minus type. The set of all directed intervals is $D = I(R) \otimes \Lambda$. For $\mathbf{A} = [a^-, a^+; \alpha] \in D$ denote $p(\mathbf{A}) = [a^-, a^+] \in I(R)$; the interval $p(\mathbf{A}) \in I(R)$ is called the *proper part* of $\mathbf{A}$. A directed interval $\mathbf{A} = [a^-, a^+; \alpha]$ is said to be degenerate if $p(\mathbf{A})$ is degenerate. Degenerate directed intervals are by definition of both plus and minus type. This means that for $a \in R$ we do not distinguish between $[a, a; +]$ and $[a, a; -]$ and write $[a, a; +] = [a, a; -] = a$. The set of all directed Z-intervals, that is directed intervals $\mathbf{A}$ with $0 \in p(\mathbf{A})$, is denoted by T. A directed interval is said to be zero-free if its proper part is a zero-free interval. $D \setminus T$ is the set of all zero-free directed intervals.

The functionals $\omega, \chi, \sigma, \phi, \psi$ from section 2 are extended for directed intervals $\mathbf{A} = [A; \alpha]$ by setting $f(\mathbf{A}) = f(p(\mathbf{A})) = f(A)$, for $f \in \{\omega, \chi, \sigma\}$. Operations between directed intervals are introduced as follows.

Addition of two directed intervals $\mathbf{A} = [a^-, a^+; \alpha], \mathbf{B} = [b^-, b^+; \beta] \in D$ is defined by

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= [a^-, a^+; \alpha] + [b^-, b^+; \beta] \\ &= \begin{cases} [a^- + b^-, a^+ + b^+; \alpha], & \text{if } \alpha = \beta, \\ [a^- + b^+, a^+ + b^-; \alpha], & \text{if } \alpha = -\beta,\ a^- + b^+ \leq a^+ + b^-, \\ [a^+ + b^-, a^- + b^+; \beta], & \text{if } \alpha = -\beta,\ a^- + b^+ > a^+ + b^-, \end{cases} \\ &= \begin{cases} [a^- + b^-, a^+ + b^+; \alpha], & \text{if } \alpha = \beta, \\ [a^{-\gamma} + b^{\gamma}, a^{\gamma} + b^{-\gamma}; \alpha\gamma], & \text{if } \alpha = -\beta, \end{cases} \end{aligned} \tag{10}$$

wherein $\gamma = \mathrm{sign}((a^+ + b^-) - (a^- + b^+)) = \phi([a^-, a^+], [b^-, b^+]) = \phi(A, B)$.

Multiplication of two directed zero-free intervals is defined by

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= [a^-, a^+; \alpha] \times [b^-, b^+; \beta] \\ &= \begin{cases} [a^{-\sigma(B)} b^{-\sigma(A)}, a^{\sigma(B)} b^{\sigma(A)}; \alpha], & \text{if } \alpha = \beta, \\ [a^{\sigma(B)} b^{-\sigma(A)}, a^{-\sigma(B)} b^{\sigma(A)}; \alpha], & \text{if } \alpha = -\beta,\ a^{\sigma(B)} b^{-\sigma(A)} \leq a^{-\sigma(B)} b^{\sigma(A)} \\ [a^{-\sigma(B)} b^{\sigma(A)}, a^{\sigma(B)} b^{-\sigma(A)}; \beta], & \text{if } \alpha = -\beta,\ a^{\sigma(B)} b^{-\sigma(A)} > a^{-\sigma(B)} b^{\sigma(A)} \end{cases} \\ &= \begin{cases} [a^{-\sigma(B)} b^{-\sigma(A)}, a^{\sigma(B)} b^{\sigma(A)}; \alpha], & \text{if } \alpha = \beta, \\ [a^{\varepsilon\sigma(B)} b^{-\varepsilon\sigma(A)}, a^{-\varepsilon\sigma(B)} b^{\varepsilon\sigma(A)}; \alpha\varepsilon], & \text{if } \alpha = -\beta. \end{cases} \end{aligned} \tag{11}$$

wherein $\varepsilon = \operatorname{sgn}(a^{-\sigma(B)}b^{\sigma(A)} - a^{\sigma(B)}b^{-\sigma(A)}) = \chi(A, B)$.

Using the form $[A; \alpha], [B; \beta]$ for the operands $\mathbf{A}$, resp. $\mathbf{B}$, we can express the sum by

$$[A; \alpha] + [B; \beta] = \begin{cases} [A + B; \alpha], & \text{if } \alpha = \beta, \\ [A +^{-} B; \alpha\gamma], & \text{if } \alpha = -\beta, \end{cases}$$

wherein $\gamma = \phi(A, B)$. In a concised form we can write

$$[A; \alpha] + [B; \beta] = [A +^{\alpha\beta} B; \tau_1([A; \alpha], [B; \beta])],$$

or, respectively,

$$\mathbf{A} + \mathbf{B} = [A +^{\alpha\beta} B; \tau_1(\mathbf{A}, \mathbf{B})],$$

wherein $\tau_1([A; \alpha], [B; \beta]) = \tau_1(\mathbf{A}, \mathbf{B})$ is defined by

$$\tau_1([A; \alpha], [B; \beta]) = \begin{cases} \alpha, & \text{if } \omega(A) \geq \omega(B), \\ \beta, & \text{if } \omega(A) < \omega(B). \end{cases}$$

Similarly we have for $A, B \in I(R) \setminus Z$

$$[A; \alpha] \times [B; \beta] = [A \times^{\alpha\beta} B; \tau_2([A; \alpha], [B; \beta])],$$

or equivalently for $\mathbf{A}, \mathbf{B} \in D \setminus T$

$$\mathbf{A} \times \mathbf{B} = [A \times^{\alpha\beta} B; \tau_2(\mathbf{A}, \mathbf{B})],$$

wherein τ_2 is given by

$$\tau_2([A; \alpha], [B; \beta]) = \begin{cases} \alpha, & \text{if } \chi(A) \geq \chi(B), \\ \beta, & \text{if } \chi(A) < \chi(B). \end{cases}$$

According to (11) multiplication by a degenerate interval is expressed by

$$a \times [b^-, b^+; \beta] = [ab^{-\sigma(a)}, ab^{\sigma(a)}; \beta].$$

If $a = -1$ we have $(-1) \times [b^-, b^+; \beta] = -[b^-, b^+; \beta] = [-b^+, -b^-; \beta]$, resp. $-[B; \beta] = [-B; \beta]$, which is called the negative of $[b^-, b^+; \beta]$. Negation preserves the type of an interval.

The inverse additive of $[a^-, a^+; \alpha]$ is the directed interval $[-a^+, -a^-; -\alpha]$. Indeed, using (10) we have:

$$[a^-, a^+; \alpha] + [-a^+, -a^-; -\alpha] = [0, 0; \pm] = 0.$$

The inverse additive is of opposite type. The inverse additive of the negative of a directed interval $[a^-, a^+; \alpha]$ is the interval $[a^-, a^+; -\alpha]$ called conjugation of $[a^-, a^+; \alpha]$; conjugation inverts the type and preserves the proper part; it is denoted by $i([a^-, a^+; \alpha]) = [a^-, a^+; \alpha]_- = [a^-, a^+; -\alpha]$, resp. $[A; \alpha]_- = [A; -\alpha]$.

Similarly, the inverse element of $[a^-, a^+; \alpha]$ with respect to the operation $\times$ is the directed interval $[1/a^+, 1/a^-; -\alpha]$; indeed we have

$$[a^-, a^+; \alpha] \times [1/a^+, 1/a^-; -\alpha] = [1, 1; \pm] = 1.$$

We write for the inverse multiplication element:

$$1/[a^-, a^+; \alpha] = [1/a^+, 1/a^-; -\alpha].$$

Subtraction of two directed intervals is defined resp. by $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$. Division of two zero-free intervals is defined by $\mathbf{A}/\mathbf{B} = \mathbf{A} \times (1/\mathbf{B})$.

The algebraic structure $(D, +, \times)$ is a rich algebraic structure. It is equivalent to the algebraic structure $(\mathcal{H}, +, \times)$, where $\mathcal{H} \cong R^2$ is the set of all ordered couples of real numbers (see [6]–[9], [17], [15]). The following associative and distributive laws hold true in $(\mathcal{H}, +, \times)$ and consequently in $(D, +, \times)$:

Proposition 2. For $\mathbf{A}, \mathbf{B}, \mathbf{C} \in D$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

Proposition 3. For $\mathbf{A}, \mathbf{B}, \mathbf{C} \in D \setminus T$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

Proposition 4. For $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{A} + \mathbf{B} \in D \setminus T$

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \begin{cases} (\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \times \mathbf{C}), & \text{if } \sigma(A) = \sigma(B) \ \ (= \sigma(A + B)), \\ (\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \times \mathbf{C}_-), & \text{if } \sigma(A) = -\sigma(B) = \sigma(A + B), \\ (\mathbf{A} \times \mathbf{C}_-) + (\mathbf{B} \times \mathbf{C}), & \text{if } \sigma(A) = -\sigma(B) = -\sigma(A + B). \end{cases}$$

We omit the verification of the above propositions which can be done directly from the definition.

Each relation between directed intervals implies a corresponding relation between the proper part of these intervals, that is a relation between normal intervals. We shall demonstrate this on the example of Proposition 2.

Substituting $\mathbf{A} = [A; \alpha], \mathbf{B} = [B; \beta], \mathbf{C} = [C; \gamma]$ in $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ we obtain:

$[A +^{\alpha\beta} B; \tau_1(\mathbf{A}, \mathbf{B})] + [C; \gamma] = [A; \alpha] + [B +^{\beta\gamma} C; \tau_1(\mathbf{B}, \mathbf{C})]$.

Comparing the proper parts of both sides we obtain for $A, B, C \in I(R)$, $\alpha, \beta, \gamma \in \Lambda$

$(A +^{\alpha\beta} B) +^{\tau_1(\mathbf{A},\mathbf{B})\gamma} C = A +^{\alpha\tau_1(\mathbf{B},\mathbf{C})} (B +^{\beta\gamma} C)$.

This equality presents the associative law for the operations $+, +^-$. Using this equality one can exchange the order of the operations in any expression involving two additions (standard and/or nonstandard). For a detailed form of this and other laws see [15]. We note that this techniques leads to a concise form of the results (for other forms cf. [3], [4], [18], [19]).

4 Presentation of ranges using directed intervals

Proposition 1 can be elegantly reformulated in terms of directed intervals. Let $f \in CM(X)$ and let $f[X] = [f(X); \tau(f;X)]$ be the directed range of f (see introduction). Then the following analogue of Proposition 1 holds true.

Proposition 5. For $f, g \in CM(D), X \subseteq D$:

$$f + g \in CM(X) \Longrightarrow (f+g)[X] = f[X] + g[X];$$
$$f - g \in CM(X) \Longrightarrow (f-g)[X] = f[X] - g[X]_{-}.$$

In addition to the above assumptions, if f, g do not change their sign in D, then

$$fg \in CM(X) \Longrightarrow (fg)[X] = f[X]_{\sigma(g(X))} \times g[X]_{\sigma(f(X))};$$
$$f/g \in CM(X), g(x) \neq 0 \Longrightarrow (f/g)[X] = f[X]_{\sigma(g(X))} / g[X]_{-\sigma(f(X))},$$

wherein $\sigma(f(X)) = \sigma(f[X])$ is the sign of the interval $f(X)$ (or of the directed interval $f[X]$, which is the same), that is the sign of f on X, so that $f[X]_{\sigma(f(X))} = \{f[X], \text{ if } f \geq 0;\ i(f[X]), \text{ if } f \leq 0\}$. Note that for $\mathbf{A} = [A; \alpha]$ and $\sigma \in \Lambda$ we have $\mathbf{A}_\sigma = [A; \sigma\alpha]$.

Proposition 4 is more powerful than Proposition 1 in the sense that it gives the type of the resulting interval as well.

Example 2. Let us repeat the task from Example 1 in terms of directed ranges and directed interval arithmetic. We have $\exp[-X] = [\exp(-X); -]$, $\mathrm{arctg}[X] = [\mathrm{arctg}X; +]$. Using Proposition 5 we obtain for the directed range of the function $h(x) = \exp(-x) + \mathrm{arctg}(x)$ the expression $h[X] = \exp[-X] + \mathrm{arctg}[X]$ for any $X \in I(R)$, where $h(x)$ is monotone, that is for $0 \notin X$.

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